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WHEN ARE HJB-EQUATIONS FOR CONTROL PROBLEMS WITH STOCHASTIC DELAY EQUATIONS FINITE DIMENSIONAL?

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Abstract. We consider optimal control problems where the state $X(t)$ at time t of the system is given by a stochastic differential delay equation. The growth at time t not only depends on the present value $X(t)$, but also on $X(t-\delta)$ and some sliding average of previous values. Morover, this dependence may be nonlinear. Using the dynamic programming principle we derive an associated (finite dimensional) Hamilton-Jacobi-Bellman equation for the value function of such problems. This (finite dimensional) HJB equation has solutions if and only if the coefficients satisfy a particular system of first order PDEs. We introduce viscosity solutions for the type of HJB-equations that we consider, and prove that under certain conditions, the value function is the unique viscosity solution to the HJB-equation. We also give numerical examples for two cases where the HJB-equation reduces to a finite dimensional one.

1. Introduction

We consider stochastic control problems where the system is given by a one-dimensional stochastic differential delay equation (SDDE) of the form

$$\begin{cases} dX(t) = b\big(t, X(t), Y(t), Z(t), u(t)\big)\,dt \\ \qquad\qquad +\sigma\big(t, X(t), Y(t), Z(t), u(t)\big)\,dB(t), \quad t \in (s,T], \quad T<\infty, \\ X(t) = \xi(t-s), \quad t\in[s-\delta, s], \quad \xi\in C([-\delta,0];\boldsymbol{R}), \end{cases} \tag{1.1}$$

where

$$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s)\,ds, \quad Z(t) = X(t-\delta),$$

b and σ are given functions, $\delta \geq 0$ is the constant *delay*, λ is constant, B is Brownian motion, and $u(t)$ is the control. The problem is to maximize the functional

$$J(s,\xi;u) = E^{s,\xi,u}\left[\int_s^T f(t, X(t), Y(t), u(t))\,dt + h(X(T), Y(T))\right] \tag{1.2}$$

over some class $\mathcal{U}[s,T]$ of control strategies. Here, $E^{s,\xi,u}$ denotes expectation with respect to the law of $X = X^{s,\xi,u}$, the solution of (1.1) when the system starts from ξ at time s and the control u is used. If we define the *value function*

$$\begin{cases} V(s,\xi) = \sup\limits_{u\in\mathcal{U}[s,T]} J(s,\xi;u), \quad (s,\xi)\in[0,T)\times C([-\delta,0];\boldsymbol{R}), \\ V(T,\xi) = h(X(0), Y(0)), \quad \xi\in C([-\delta,0];\boldsymbol{R}), \end{cases} \tag{1.3}$$

then the task is to find the value function $V(s,\xi)$ and the associated optimal control $\bar{u}\in\mathcal{U}[s,T]$ such that $V(s,\xi) = J(s,\xi;\bar{u})$.

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In general the value function may depend on the initial path ξ in a complicated way. From [11] we know that the value function satisfies the dynamic programming principle:

$$V(s,\xi) = \sup_{u\in\mathcal{U}[s,T]} E^{s,\xi,u}\left[\int_s^{\hat{s}} f(t,X(t),Y(t),u(t))\,dt + V(\hat{s},X_{\hat{s}}^{s,\xi,u})\right]. \tag{1.4}$$

Here, $X_{\hat{s}}$ is the map $X_{\hat{s}} : [-\delta,0]\to \boldsymbol{R}$ defined by $X_{\hat{s}}(\tau) = X(\hat{s}+\tau)$. If we define the operator $\mathcal{L}^u$ on functions $W : [0,T]\times C([-\delta,0];\boldsymbol{R})\to\boldsymbol{R}$ by

$$\mathcal{L}^u W(s,\xi) := \lim_{\hat{s}\to s^+} \frac{E\big[W(\hat{s},X_{\hat{s}}^{s,\xi,u})\big] - W(s,\xi)}{\hat{s}-s}, \tag{1.5}$$

then by (1.4) the value function solves the backward evolution equation

$$\begin{cases} \inf_{u\in U}\big\{-\mathcal{L}^u V(s,\xi) - f(s,\xi,u)\big\} = 0 \quad \text{for } (s,\xi)\in[0,T)\times C([-\delta,0];\boldsymbol{R}),\\ V(T,\xi) = h(X(0),Y(0)), \end{cases} \tag{1.6}$$

where $U\subset\boldsymbol{R}$ is a given set of control *values*. Thus the problem is infinite dimensional. However, looking at the functional (1.2) one might expect that the value function depends on ξ only through the two functionals

$$x = x(\xi) := \xi(0) \quad\text{and}\quad y = y(\xi) := \int_{-\delta}^0 e^{\lambda s}\xi(s)\,ds.$$

We show that if this is the case, then the operator (1.5) is a differential operator and equation (1.6) is a second order Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE). Moreover, we show that the value function is the unique viscosity solution of this HJB-equation. This is essentially due to the fact that there is an Ito-formula in this context.

Since the coefficients of (1.1) enters into the Ito formula, the coefficients of the HJB-equation also depend on a third functional of the initial path, namely

$$z = z(\xi) := \xi(-\delta).$$

Consequently, we cannot apriori expect the HJB-equation to have solutions independent of z.

We then raise two questions: 1) If we can find a solution to the HJB-equation independent of z, will it be the value function? 2) Under what conditions will the HJB-equation have a solution depending only on (s,x,y)?

Regarding 1): If the HJB-equation has a *smooth* solution independent of z, then by a verification theorem in [5] this must be the value function. For viscosity solutions, the question is to the best of our knowledge still open.

Regarding 2): Reasoning formally as in the smooth case, we show that if the system (1.1) is on the form

$$\begin{cases} dX = \big[\mu(X,Y) + \beta(X,Y)Z - g(t,X,Y,u)\big]dt + \sigma(X,Y)dB, \quad t\in(s,T],\\ X(t) = \xi(t-s), \quad t\in[s-\delta,s], \quad \xi\in C([-\delta,0];\boldsymbol{R}), \end{cases}$$

then the HJB-equation has a solution depending only on (s,x,y) provided an auxiliary system of four first order PDEs involving μ, β, g, σ, f, and h has a solution. When this is the case, the HJB-equation reduces to an “effective” equation in only one spatial variable in addition to time.

We investigate the auxiliary system through several examples. The simplest example is the case of linear μ, g, and σ, and constant β. For this case a complete discussion is possible. We recover results on the relationships between the coefficients discovered earlier by others who have treated systems with this type of linear past dependence. Elsanousi and Larssen [5] solved a problem of optimal consumption from a linear system with delay for f and h of HARA utility type. They used a (smooth case) verification theorem to obtain classical solutions to the HJB-equation. This technique was also used by Elsanousi, Øksendal and Sulem in [4] where a singular control was used, and by Elsanousi in [3] where optimal stopping problems and impulse control problems were treated. Another approach was taken by Øksendal and Sulem in [17] where a maximum principle for optimal control of stochastic systems with delay was developed.

We also mention that Kolmanovskii and Maizenberg [9] and Kolmanovskii and Shaikhet [10] solved linear-quadratic regulator problems with delay, also by variational methods but different from ours. Other authors, such as Koivo [8] and Lindquist [12, 13] have studied variants of linear-quadratic regulator problems with delay using more direct methods exploiting the special structure of the problem.

For the nonlinear case, we are only able to solve the auxiliary system when β is a constant. Already with β linear in x and independent of y we get inconsistencies, recoverable only through a collapse of the system to the trivial non-delay case. This illustrates the difficulties with nonlinear systems and indicates why most work on the subject deals with linear systems only.

HJB-equations are nonlinear PDEs and will as such in general not have smooth solutions. The ability to characterize the value function as the unique viscosity solution of an associated HJB-equation is by now standard in stochastic optimal control theory for systems without delay, see e.g. [6], [18], or [20]. Also, numerical methods for solving HJB-equations generally converge only to a viscosity solution, see e.g. [19]. This makes a powerful framework for solving stochastic control problems in practical applications. Building on the dynamic programming principle for delay systems [11], this paper is a contribution to developing this type of framework also for delay systems.

The advantage of explicitly incorporating time delays in modeling equations is to recognize the reality of non-instantaneous interactions. For example in the population models of bioeconomics, delays may be included to represent regeneration times, maturation periods, feeding times, reaction times or take account of age structure. Much work has been done trying to understand the *dynamics* of such models, usually described by ordinary differential delay equations without controls or with constant harvesting rates, see e.g. [2], [7], and [14]. We believe that an increased understanding of control problems for delay systems could lead to better models for optimal management of renewable resources. Also see Chapter 1 in [10] for a very interesting and wider discussion of modeling systems with delay in mechanics and engineering, biology, and medicine.

We should mention that the operator $\mathcal{L}^u$ defined in (1.5) is closely related to Mohammed's concept of *weak infinitesimal generator*, see [15, 16].

The rest of the paper is organized as follows. In Section 2 we give some additional notation and assumptions. In Section 3 we show how to derive the HJB-equation. In Section 4 we define viscosity solutions and show that if the value function depends on (s, x, y) only, then it is the unique viscosity solution of the HJB-equation. In Section 5 we derive the auxiliary system of first order PDEs that the coefficients must satisfy to ensure that the HJB-equation has a solution independent of z. This is Theorem 5.1. Finally, in Section 6 we give two examples that satisfy the requirements given in Theorem 5.1, and also provide numerical solutions to the HJB-equation in each case. We also indicate why it is difficult to find more general examples.

2. Notation and assumptions

We assume B in (1.1) is a one-dimensional Brownian motion on a given complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{s\le t\le T}, \boldsymbol{P})$.

The class $\mathcal{U}[s,T]$ of admissible control *strategies* is defined as follows. First, let $U \subset \boldsymbol{R}$ be a given set of admissible control *values*. For $s \in [0,T)$ let $\mathcal{U}[s,T]$ denote the set of all 5-tuples $(\Omega, \mathcal{F}, \boldsymbol{P}, W, u)$ satisfying the following:

C1 $(\Omega, \mathcal{F}, \boldsymbol{P})$ is a complete probability space.

C2 $\{B(t)\}_{s\le t\le T}$ is a one-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \boldsymbol{P})$ over $[s,T]$ with $B(s) = 0$ a.s., and $\mathcal{F}^s_t = \sigma\{B(\tau)\,;\, s \le \tau \le t\}$ augmented by the $\boldsymbol{P}$-null sets in $\mathcal{F}$.

C3 $u : [s,T] \times \Omega \to U$ is an $\{\mathcal{F}^s_t\}$-adapted process on $(\Omega, \mathcal{F}, \boldsymbol{P})$.

C4 Under u, for any $\varphi \in C[-\delta, 0]^n$ equation (1.1) admits a unique strong solution X on $(\Omega, \mathcal{F}, \{\mathcal{F}^s_t\}, \boldsymbol{P})$.

C5 The process u is such that $E^{s,\xi,u}\left[\int_s^T |f(t, X(t), Y(t), u(t))|\, dt + |h(X(T), Y(T))|\right] < \infty$

We usually write $u \in \mathcal{U}[s,T]$ instead of $(\Omega, \mathcal{F}, P, W, u) \in \mathcal{U}[s,T]$. Then the control problem can be stated as

Problem 2.1. For given $(s,\xi) \in [0,T) \times C([-\delta,0]; R)$, find a 5-tuple $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \bar{B}, \bar{u}) \in \mathcal{U}[s,T]$ such that

$$J(s,\xi;\bar{u}) = \sup_{u \in \mathcal{U}[s,T]} J(s,\xi;u) \tag{2.1}$$

In order to ensure that the SDDE (1.1) has a unique solution we introduce the following assumptions. Let $\eta = (x,y) \in R^2$.

A1 The maps $b(t,\eta,u)$ and $\sigma(t,\eta,u)$ are globally Lipschitz in the second variable, that is, there is a constant $L > 0$ such that

$$|b(t,\eta,u) - b(t,\hat{\eta},u)| + |\sigma(t,\eta,u) - \sigma(t,\hat{\eta},u)| \le L\,|\eta - \hat{\eta}|$$

for all $0 \le t \le T$, η, $\hat{\eta} \in R^2$, and $u \in U$.

A2 The maps $b : R^4 \times U \to R$ and $\sigma : R^4 \times U \to R$ are continuous.

A3 The initial path ξ belongs to the space $L^2(\Omega; C([-\delta,0]; R); \mathcal{F}_s^s)$ of $\mathcal{F}_s^s$-measurable elements in $L^2(\Omega; C([-\delta,0]; R))$ that is, $\xi : \Omega \to C([-\delta,0]; R)$ is $\mathcal{F}_s^s$-measurable and

$$\|\xi\|^2_{L^2(\Omega; C([-\delta,0]; R))} := E\big[\|\xi(\omega)\|^2_{C([-\delta,0];R)}\big] < \infty.$$

A4 The maps $f : [0,T] \times R \times U \to R$ and $h : R^2 \to R$ are uniformly continuous, and there exists a constant $L > 0$ such that

$$|f(t,\eta,u) - f(t,\hat{\eta},u)| + |h(\eta) - h(\hat{\eta})| \le L\,|\eta - \hat{\eta}|\,,$$

for all $t \in [0,T]$, $\eta, \hat{\eta} \in C([-\delta,0]; R)$, $u \in U$, and

$$|f(t,0,u)| + |h(0)| \le L, \quad \text{for all } (t,u) \in [0,T] \times U.$$

Theorem 2.1. *Under the assumptions A1–A3, for any* $(s,\xi) \in [0,T) \times L^2(\Omega; C([-\delta,0]; R))$ *and* $u \in \mathcal{U}[s,T]$*, the SDDE* (1.1) *admits a unique adapted strong solution*

$$X = X^{s,\xi,u} \in L^2(\Omega; C([-\delta,T]; R); \mathcal{F}_t^s)).$$

Proof. This is due to Mohammed and follows from Theorem I.2 in [16] or Theorem 2.1 in [15]. □

Under the assumption A4 the performance functional (1.2) and the value function (1.3) are well-defined.

3. The Hamilton-Jacobi-Bellman equation

In general the value function may depend on the initial path $\xi \in C([-\delta,0]; R)$ in a complicated way. Never the less, in [5] it is shown that for certain choices of the functions b, σ, f, and h the value function depends on the initial path only through the functionals $x(\xi)$ and $y(\xi)$, that is,

$$V(s,\xi) = V(s,x(\xi),y(\xi)) = V(s,x,y). \tag{3.1}$$

The paper [11] establishes that the value function satisfies the dynamic programming principle (DPP). Assuming that (3.1) holds, the DPP takes the form

$$V(s,x,y) = \sup_{u \in \mathcal{U}[s,T]} E^{s,\xi,u}\Bigg[\int_s^{s+d} f(t,X(t),Y(t),u(t))\,dt + V(s+d, X(s+d), Y(s+d))\Bigg] \tag{3.2}$$

for all $d \in [0, T-s)$ and $(x,y) \in R^2$ where $\xi \in C([-\delta,0]; R)$ is such that $x = x(\xi) = X(s)$ and $y = y(\xi) = Y(s)$. Also, for systems like (1.1) there exists an Ito formula. Using this and the DPP, we shall see that if the value function satisfies (3.1) and is smooth enough, it solves a Hamilton-Jacobi-Bellman type PDE. To state the Ito formula, let $g \in C^{1,2,1}(R^3)$ and define

$$G(t) = g(t,X(t),Y(t)). \tag{3.3}$$

Lemma 3.1 (The Ito formula)**.**

$$dG(t) = \frac{\partial}{\partial t} g(t,x,y)\,dt + A_z^u g(t,x,y)\,dt + (x - e^{-\lambda\delta} z - \lambda y)\frac{\partial}{\partial y} g(t,x,y)\,dt \\ + \sigma(t,x,y,z,u)\frac{\partial}{\partial x} g(t,x,y)\,dB(t) \tag{3.4}$$

where

$$A_z^u g(t,x,y) := b(t,x,y,z,u)\frac{\partial}{\partial x} g(t,x,y) + \frac{1}{2}\sigma^2(t,x,y,z,u)\frac{\partial^2}{\partial x^2} g(t,x,y). \tag{3.5}$$

All the expressions are evaluated at

$$x = X(t), \quad y = Y(t), \quad z = Z(t).$$

Proof. See [4] or [5]. □

Lemma 3.2 (The Hamilton-Jacobi-Bellman equation)**.** If the value function V depends on (s,x,y) only and $V(s,x,y) \in C^{1,2,1}([0,T] \times \boldsymbol{R}^2)$ then $V(s,x,y)$ solves the following Hamilton-Jacobi-Bellman (HJB) partial differential equation

$$-\frac{\partial}{\partial s} V(s,x,y) + \inf_{u \in U}\big\{-A_z^u V(s,x,y) - f(s,x,y,u)\big\} \\ - (x - e^{-\lambda\delta} z - \lambda y)\frac{\partial}{\partial y} V(s,x,y) = 0 \quad \text{for all } z \in \boldsymbol{R}, \tag{3.6}$$

with terminal condition

$$V(T,x,y) = h(x,y). \tag{3.7}$$

Proof. Fix $(s,x,y) \in [0,T) \times \boldsymbol{R}^2$, $d \in (0, T-s)$, and $u \in U$ (u is a control *value*). Let X be given by (1.1) with $u(t) \equiv u$, and fix $\xi \in C[-\delta, 0]$ such that $x = x(\xi) = X(s)$ and $y = y(\xi) = Y(s)$. Then (3.2) implies that

$$V(s,x,y) \geq \boldsymbol{E}^{s,\xi,u}\left[\int_s^{s+d} f(t, X(t), Y(t), u(t))\,dt + V(s+d, X(s+d), Y(s+d))\right].$$

Dividing by d and rearranging we see that

$$\frac{\boldsymbol{E}^{s,\xi,u}\big[V(s+d, X(s+d), Y(s+d))\big] - V(s,x,y)}{d} \\ + \boldsymbol{E}^{s,\xi,u}\left[\frac{1}{d}\int_s^{s+d} f(t, X(t), Y(t), u(t))\,dt\right] \leq 0.$$

Now use the Ito formula on $V(s+d, X(s+d), Y(s+d))$ and note that the dB-integral in (3.4) has zero expectation. As $d \to 0$ we obtain,

$$\frac{\partial}{\partial s} V(s,x,y) + A_z^u V(s,x,y) + f(s,x,y,u) \\ + (x - e^{-\lambda\delta} z - \lambda y)\frac{\partial}{\partial y} V(s,x,y) \leq 0 \quad \text{for all } z \in \boldsymbol{R}.$$

This holds for any $u \in U$, so

$$\frac{\partial}{\partial s} V(s,x,y) + \sup_{u \in U}\big\{A_z^u V(s,x,y) + f(s,x,y,u)\big\} \\ + (x - e^{-\lambda\delta} z - \lambda y)\frac{\partial}{\partial y} V(s,x,y) \leq 0 \quad \text{for all } z \in \boldsymbol{R},$$

or equivalently,

$$
(3.8)\quad -\frac{\partial}{\partial s}V(s,x,y)+\inf_{u\in U}\bigl\{-A_z^u V(s,x,y)-f(s,x,y,u)\bigr\} \\
-(x-e^{-\lambda\delta}z-\lambda y)\frac{\partial}{\partial y}V(s,x,y)\geq 0 \quad \text{for all } z\in \boldsymbol{R}.
$$

Conversely, for any $\varepsilon>0$, $0\leq s\leq T$, with $d>0$ small enough, we can find $u(\cdot)=u_{\varepsilon,d}(\cdot)\in \mathcal{U}[s,T]$ such that

$$
V(s,x,y)-\varepsilon d\leq \boldsymbol{E}^{s,\xi,u}\left[\int_s^{s+d} f(t,X(t),Y(t),u(t))\,dt+V(s+d,X(s+d),Y(s+d))\right],
$$

or equivalently,

$$
-\varepsilon\leq \frac{\boldsymbol{E}^{s,\xi,u}\bigl[V(s+d,X(s+d),Y(s+d))\bigr]-V(s,x,y)}{d} \\
+\boldsymbol{E}^{s,\xi,u}\left[\frac{1}{d}\int_s^{s+d} f(t,X(t),Y(t),u(t))\,dt\right].
$$

Again, as $d\to 0$ we see that

$$
\begin{aligned}
-\varepsilon &\leq \frac{\partial}{\partial s}V(s,x,y)+A_z^u V(s,x,y)+f(s,x,y,u) \\
&\qquad +(x-e^{-\lambda\delta}z-\lambda y)\frac{\partial}{\partial y}V(s,x,y) \\
&\leq \frac{\partial}{\partial s}V(s,x,y)+\sup_{u\in U}\bigl\{A_z^u V(s,x,y)+f(s,x,y,u)\bigr\} \\
&\qquad +(x-e^{-\lambda\delta}z-\lambda y)\frac{\partial}{\partial y}V(s,x,y) \quad \text{for all } z\in\boldsymbol{R},
\end{aligned}
$$

or equivalently,

$$
(3.9)\quad \varepsilon\geq -\frac{\partial}{\partial s}V(s,x,y)+\inf_{u\in U}\bigl\{-A_z^u V(s,x,y)-f(s,x,y,u)\bigr\} \\
-(x-e^{-\lambda\delta}z-\lambda y)\frac{\partial}{\partial y}V(s,x,y) \quad \text{for all } z\in \boldsymbol{R}.
$$

Combining the inequalities (3.8) and (3.9) gives (3.6). The terminal condition (3.7) follows immediately from the definition of the value function. □

4. Viscosity solutions

The following is adapted from Soner's C.I.M.E. lectures [18] and the book [20] by Yong and Zhou.

Definition 4.1. Let W be a continuous function on $[0,T]\times \boldsymbol{R}^2$.

Subsolution: We say that W is a *viscosity subsolution* of (3.6)–(3.7) in $[0,T]\times\boldsymbol{R}^2$ if

$$
(4.1)\qquad W(T,x,y)\leq h(x,y) \quad \text{for all } (x,y)\in\boldsymbol{R}^2,
$$

and for any $\varphi\in C^\infty((0,T)\times\boldsymbol{R}^2)$, whenever $W-\varphi$ attains a local maximum at $(s,x,y)\in(0,T)\times\boldsymbol{R}^2$, we have

$$
(4.2)\quad -\frac{\partial}{\partial s}\varphi(s,x,y)+\inf_{u\in U}\bigl\{-A_z^u\varphi(s,x,y)-f(s,x,y,u)\bigr\} \\
-(x-e^{-\lambda\delta}z-\lambda y)\frac{\partial}{\partial y}\varphi(s,x,y)\leq 0 \quad \text{for all } z\in\boldsymbol{R}.
$$

Supersolution: We say that W is a *viscosity supersolution* of (3.6)–(3.7) in $[0,T]\times \boldsymbol{R}^2$ if

$$W(T,x,y) \geq h(x,y) \quad \text{for all } (x,y)\in \boldsymbol{R}^2, \tag{4.3}$$

and for any $\varphi \in C^\infty((0,T)\times \boldsymbol{R}^2)$, whenever $W-\varphi$ attains a local minimum at $(s,x,y)\in (0,T)\times \boldsymbol{R}^2$, we have

$$\begin{aligned} -\frac{\partial}{\partial s}\varphi(s,x,y) + \inf_{u\in U}\{-A_z^u\varphi(s,x,y) - f(s,x,y,u)\} \\ -(x-e^{-\lambda\delta}z-\lambda y)\frac{\partial}{\partial y}\varphi(s,x,y) \geq 0 \quad \text{for all } z\in \boldsymbol{R}. \end{aligned} \tag{4.4}$$

Solution: We say that W is a *viscosity solution* of (3.6) in $[0,T]\times \boldsymbol{R}^2$ if it is both a viscosity subsolution and a viscosity supersolution in $[0,T]\times\boldsymbol{R}^2$.

Theorem 4.1. *Under the assumptions* (A1)–(A4), *if the value function depends on* (s,x,y) *only, then it is the unique viscosity solution of the HJB-equation* (3.6).

Proof. It is shown in [11] that the value function is continuous. We first show that V is a *subsolution*: Let $\varphi\in C^\infty((0,T)\times\boldsymbol{R}^2)$, and assume $(s,x,y)\in[0,T)\times\boldsymbol{R}^2$ is a local minimum of $V-\varphi$. Fix $\xi\in C[-\delta,0]$ such that $x=x(\xi)=X(s)$ and $y=y(\xi)=Y(s)$. Then for $d\in(0,T-s)$ small enough

$$\begin{aligned} 0 \leq V(s,x,y) - V(s+d,X(s+d),Y(s+d)) \\ + \varphi(s+d,X(s+d),Y(s+d)) - \varphi(s,x,y). \end{aligned} \tag{4.5}$$

The DPP implies that for any $\varepsilon>0$ with $d>0$ small enough, there exists a control $u=u_{\varepsilon,d}\in \mathcal{U}[s,T]$ such that

$$V(s,x,y)-\varepsilon d \leq \boldsymbol{E}^{s,\xi,u}\left[\int_s^{\hat{s}} f(t,X(t),Y(t),u(t))\,dt + V(s+d,X(s+d),Y(s+d))\right],$$

which combined with (4.5) gives

$$\begin{aligned} 0 \leq \varepsilon d + \boldsymbol{E}^{s,\xi,u}\Bigg[\int_s^{\hat{s}} f(t,X(t),Y(t),u(t))\,dt \\ + \varphi(s+d,X(s+d),Y(s+d)) - \varphi(s,x,y)\Bigg]. \end{aligned}$$

Dividing by d, using the Ito formula and letting $d\to 0$ we see that

$$\begin{aligned} -\varepsilon &\leq \frac{\partial}{\partial s}\varphi(s,x,y) + A_z^u\varphi(s,x,y) + f(s,x,y,u) \\ &\qquad + (x-e^{-\lambda\delta}z-\lambda y)\frac{\partial}{\partial y}\varphi(s,x,y) \\ &\leq \frac{\partial}{\partial s}\varphi(s,x,y) + \sup_{u\in U}\{A_z^u\varphi(s,x,y) + f(s,x,y,u)\} \\ &\qquad + (x-e^{-\lambda\delta}z-\lambda y)\frac{\partial}{\partial y}\varphi(s,x,y) \quad \text{for all } z\in\boldsymbol{R}, \end{aligned}$$

or equivalently,

$$\begin{aligned} \varepsilon \geq -\frac{\partial}{\partial s}\varphi(s,x,y) + \inf_{u\in U}\{-A_z^u\varphi(s,x,y) - f(s,x,y,u)\} \\ -(x-e^{-\lambda\delta}z-\lambda y)\frac{\partial}{\partial y}\varphi(s,x,y) \quad \text{for all } z\in\boldsymbol{R}. \end{aligned} \tag{4.6}$$

This holds for all $\varepsilon>0$, hence φ is a subsolution.

Next, we demonstrate that V is also a *supersolution*: Let $\varphi\in C^\infty((0,T)\times\boldsymbol{R}^2)$ be a smooth test function, and assume $(s,x,y)\in[0,T)\times\boldsymbol{R}^2$ is a local minimum of $V-\varphi$. Fix $\xi\in C[-\delta,0]$

such that $x = x(\xi) = X(s)$ and $y = y(\xi) = Y(s)$, and $u \in U$ (u is a control *value*). Let X be given by (1.1) with $u(t) \equiv u$. Now for $d \in (0, T - s)$ small enough, we have

$$0 \ge V(s,x,y) - V(s+d, X(s+d), Y(s+d)) \\ + \varphi(s+d, X(s+d), Y(s+d)) - \varphi(s,x,y),$$

or

$$\begin{aligned} 0 &\ge \frac{1}{d} E^{s,\xi,u}\big[V(s,x,y) - V(s+d, X(s+d), Y(s+d)) \\ &\qquad + \varphi(s+d, X(s+d), Y(s+d)) - \varphi(s,x,y)\big] \\ &\ge \frac{1}{d} E^{s,\xi,u}\left[\int_s^{s+d} f(t, X(t), Y(t), u)\,dt + \varphi(s+d, X(s+d), Y(s+d)) - \varphi(s,x,y)\right], \end{aligned}$$

where the last inequality is due to the DPP. Using the Ito formula on $\varphi(s+d, X(s+d), Y(s+d))$ this is equivalent to

$$0 \ge \frac{1}{d} E^{s,\xi,u}\left[\int_s^{s+d} \Big\{ f(t, X(t), Y(t), u)\,dt + \frac{\partial}{\partial t}\varphi(t, X(t), Y(t)) + A^u_{Z(t)}\varphi(t, X(t), Y(t)) \\ + [X(t) - e^{-\lambda\delta} Z(t) - \lambda Y(t)] \frac{\partial}{\partial y}\varphi(t, X(t), Y(t)) \Big\}\,dt\right],$$

and as $d \to 0$ we obtain

$$\frac{\partial}{\partial s}\varphi(s,x,y) + A^u_z\varphi(s,x,y) + f(s,x,y,u) \\ + (x - e^{-\lambda\delta} z - \lambda y)\frac{\partial}{\partial y}\varphi(s,x,y) \le 0 \quad \text{for all } z \in \boldsymbol{R}.$$

By the proof of Lemma 3.2 this implies

$$-\frac{\partial}{\partial s}\varphi(s,x,y) + \inf_{u\in U}\big\{-A^u_z\varphi(s,x,y) - f(s,x,y,u)\big\} \\ - (x - e^{-\lambda\delta} z - \lambda y)\frac{\partial}{\partial y}\varphi(s,x,y) \le 0 \quad \text{for all } z \in \boldsymbol{R},$$

which shows that V is a supersolution.

The uniqueness follows from general theory, see e.g. [1].

□

5. When is the solution of the HJB-equation independent of z?

In the last section we saw that if the value function depends on (s, x, y) only, then it is a viscosity solution of the HJB equation (3.6). In this section we start at the other end. We take the HJB equation as given and seek conditions ensuring that a solution will be independent of z. We obtain a result for a system less general than (1.1), but which never the less covers many interesting applications.

To start, we use the setup from the previous sections and define

$$F = F(s,x,y,z,u,DV,D^2V) := -A^u_z V(s,x,y) - f(s,x,y,u), \tag{5.1}$$

and

$$\ell(x,y,z) = x - e^{-\lambda\delta} z - \lambda y. \tag{5.2}$$

Then the HJB-equation (3.6) takes the form

$$-\partial_s V + \inf_{u\in U} F - \ell\partial_y V = 0 \quad \text{for all } z \in \boldsymbol{R}. \tag{5.3}$$

Assume that $\bar{u}$ is an optimal control and put $\bar{F} = F(\bar{u})$. Then

$$-\partial_s V + \bar{F} - \ell\partial_y V = 0 \quad \text{for all } z \in \boldsymbol{R}. \tag{5.4}$$

Since this holds for all z, we must have $\partial_z(\bar{F} - \ell\partial_y V) = 0$, that is,

$$\partial_{\bar{u}}\bar{F} \cdot \bar{u}_z + \partial_z \bar{F} - \ell_z \partial_y V = 0. \tag{5.5}$$

Now $\partial_{\bar{u}}\bar{F} = 0$ since $\bar{u}$ is optimal. With $\ell_z = -e^{-\lambda\delta}$ this leads to $\partial_z\bar{F} + e^{-\lambda\delta}\partial_y V = 0$ or

$$\partial_y V = -e^{\lambda\delta}\partial_z\bar{F} \tag{5.6}$$

which we insert into (5.4) to obtain

$$-\partial_s V + \bar{F} + \ell e^{\lambda\delta}\partial_z\bar{F} = 0. \tag{5.7}$$

Here, $\bar{F} + \ell e^{\lambda\delta}\partial_z\bar{F}$ should not depend on z. In the following, let H and G denote generic functions that may depend on $s, x, y, \bar{u}, DV, D^2V$ but *not* on z. (H and G may change from line to line in a calculation.) Then $\bar{F} + \ell e^{\lambda\delta}\partial_z\bar{F} = H$ and the following are equivalent:

$$\begin{aligned}
\bar{F} + \ell e^{\lambda\delta}\partial_z\bar{F} &= H,\\
e^{-\lambda\delta}\bar{F} + \ell\partial_z\bar{F} &= H,\\
\partial_z\left(\frac{\bar{F}}{\ell}\right) &= \frac{\ell\partial_z\bar{F} - \ell_z\bar{F}}{\ell^2} = \frac{H}{\ell^2}.
\end{aligned}$$

Integrating this yields

$$\frac{\bar{F}}{\ell} = H\int\frac{dz}{\ell^2} = -He^{\lambda\delta}\int\frac{d\ell}{\ell^2} = \frac{-He^{\lambda\delta}}{\ell} + G,$$

so that $\bar{F} = H + G\ell$, which implies that $\bar{F}$ is linear in z, i.e.,

$$\bar{F} = H + Gz,$$

where H and G are functions that do not depend on z.

Motivated by the above reasoning, we investigate more closely a modified version of (1.1). Let X satisfy the SDDE

$$\begin{cases} dX = [\bar{\mu}(X,Y,Z) - g(t,X,Y,u)]\,dt + \bar{\sigma}(X,Y,Z)dB, \quad t \in (s,T],\\ X(t) = \xi(t-s), \quad t \in [s-\delta, s], \quad \xi \in C([-\delta,0];\boldsymbol{R}). \end{cases} \tag{5.8}$$

Recall the performance functional

$$J(s,\xi;u) = E^{s,\xi,u}\left[\int_s^T f(t,X(t),Y(t),u(t))\,dt + h(X(T),Y(T))\right], \tag{5.9}$$

and the value function

$$\begin{cases} V(s,\xi) = \sup\limits_{u\in\mathcal{U}[s,T]} J(s,\xi;u), \quad (s,\xi) \in [0,T) \times C([-\delta,0];\boldsymbol{R}),\\ V(T,\xi) = h(x(\xi),y(\xi)), \quad \xi \in C([-\delta,0];\boldsymbol{R}). \end{cases} \tag{5.10}$$

We know that if $V = V(s,x,y)$, then V satisfies the HJB equation

$$-\partial_s V - \bar{\mu}\partial_x V - \frac{1}{2}\bar{\sigma}^2\partial_x^2 V - \left(x - e^{-\lambda\delta}z - \lambda y\right)\partial_y V + F\left(\partial_x V, x, y, s\right) = 0, \tag{5.11}$$

with terminal condition

$$V(T,x,y) = h(x,y), \tag{5.12}$$

where

$$F(p,x,y,s) = \inf_u\big\{(g(s,x,y,u)p - f(s,x,y,u)\big\}. \tag{5.13}$$

We wish to obtain conditions on $\bar{\mu}$, $\bar{\sigma}$ and F that ensure that (5.11) has a solution independent of z. Differentiating (5.11) with respect to z we obtain

$$\partial_y V - e^{\lambda\delta}\partial_z\bar{\mu}\partial_x V = e^{\lambda\delta}\partial_z\bar{\gamma}\partial_x^2 V, \tag{5.14}$$

where $\bar\gamma = \bar\sigma^2/2$. Inserting this into (5.11), this equation now takes the form

$$\begin{aligned}-\partial_t V &- \left[\bar\mu - (z - e^{\lambda\delta}(x-\lambda y))\partial_z \bar\mu\right]\partial_x V \\ &- \left[\bar\gamma - (z - e^{\lambda\delta}(x-\lambda y))\partial_z \bar\gamma\right]\partial_x^2 V + F\left(\partial_x V, x, y, t\right) = 0.\end{aligned}$$

If V is to be independent of z, then the coefficients of $\partial_x V$ and $\partial_x^2 V$ must be independent of z. By arguments analogous to the previous, we see that

$$\bar\mu(x,y,z) = \mu(x,y) + \beta(x,y)z,$$

and

$$\bar\gamma(x,y,z) = \gamma(x,y) + \zeta(x,y)z,$$

for some functions μ, β, γ, and ζ depending on x and y only. Now, since $\bar\gamma \geq 0$ for all (x,y,z), we must have $\zeta = 0$, and consequently $\partial_z\bar\gamma = 0$. Also note that $\partial_z \bar\mu = \beta$, and that (5.14) takes the form

$$\partial_y V - e^{\lambda\delta}\beta(x,y)\partial_x V = 0. \tag{5.15}$$

Using this in (5.11) we see that this equation now reads

$$-\partial_s V - \left[\mu(x,y) + e^{\lambda\delta}(x-\lambda y)\beta(x,y)\right]\partial_x V - \frac{1}{2}\sigma^2(x,y)\partial_x^2 V + F\left(\partial_x V, x, y, s\right) = 0. \tag{5.16}$$

Now we introduce new variables $\tilde x$ and $\tilde y$, such that

$$\frac{\partial}{\partial \tilde y} = \frac{\partial}{\partial y} - e^{\lambda\delta}\beta(x,y)\frac{\partial}{\partial x}, \quad \text{and} \quad \frac{\partial}{\partial \tilde x} = \frac{\partial}{\partial x}. \tag{5.17}$$

Then (5.15) states that $\partial_{\tilde y} V = 0$. In order to be compatible with $\partial_{\tilde y} V = 0$, the coefficients of equation (5.16) and the function h must also be constant in $\tilde y$, or

$$\partial_y \hat\mu - e^{\lambda\delta}\beta\partial_x\hat\mu = 0, \tag{5.18}$$

$$\partial_y \sigma - e^{\lambda\delta}\beta\partial_x\sigma = 0, \tag{5.19}$$

$$e^{\lambda\delta}p\partial_p F\partial_x\beta + \partial_y F - e^{\lambda\delta}\beta\partial_x F = 0, \tag{5.20}$$

$$\partial_y h - e^{\lambda\delta}\beta\partial_x h = 0, \tag{5.21}$$

where

$$\hat\mu(x,y) = \mu(x,y) + e^{\lambda\delta}(x-\lambda y)\beta(x,y).$$

To see why $\partial_{\tilde y} F = 0$ is equivalent to (5.20), note that

$$\begin{aligned}\partial_{\tilde y} F &= \partial_p F\partial_y V_x + \partial_y F - e^{\lambda\delta}\beta(\partial_p F\partial_x V_x + \partial_x F) \\ &= \partial_p F(V_{yx} - e^{\lambda\delta}\beta V_{xx}) + \partial_y F - e^{\lambda\delta}\beta\partial_x F \\ &= \partial_p F[\partial_x(V_y - e^{\lambda\delta}\beta V_x) + e^{\lambda\delta}\partial_x\beta V_x] + \partial_y F - e^{\lambda\delta}\beta\partial_x F \\ &= e^{\lambda\delta}p\partial_p F\partial_x\beta + \partial_y F - e^{\lambda\delta}\beta\partial_x F \qquad \text{by (5.15)}.\end{aligned}$$

Conversely, if $\bar\mu = \mu(x,y) + \beta(x,y)z$ and $\bar\sigma = \sigma(x,y)$, and (5.18)–(5.20) holds, then we can find a solution of (5.11) that is independent of z.

We collect this in the following theorem.

Theorem 5.1. *The HJB-equation* (5.11)–(5.12) *has a viscosity solution* $W = W(s,x,y)$ *if and only if* $\bar\mu = \mu(x,y) + \beta(x,y)z$ *and* $\bar\sigma = \sigma(x,y)$, *and* (5.18)–(5.21) *holds. In this case, in the coordinates given by* (5.17)*, the HJB equation* (5.11)–(5.12) *reads*

$$-\partial_s V - \hat\mu(\tilde x)\partial_{\tilde x} V - \frac{1}{2}\sigma^2(\tilde x)\partial_{\tilde x}^2 V + F\left(\partial_{\tilde x} V, \tilde x, s\right) = 0, \tag{5.22}$$

with the terminal condition

$$V(T,\tilde x) = h(\tilde x). \tag{5.23}$$

Remark 5.1. Recall that in general the value function $V = V(s,\xi)$ where $\xi \in C([-\delta,0];\boldsymbol{R})$. This means that we may have $W \neq V$, that is, the solution provided by the above theorem is not necessarily the value function. To be able to conclude that $W = V$, we need a *verification theorem* for viscosity solutions. A verification theorem essentially says that if we can find a viscosity solution to the HJB-equation, then this must be the value function (which then depends on (s,x,y) only). By Theorem 5.1 we can only hope to achieve this when the system is on the form

$$\begin{cases} dX = \big[\mu(X,Y) + \beta(X,Y)Z - g(t,X,Y,u)\big]dt + \sigma(X,Y)dB, \quad t \in (s,T], \\ X(t) = \xi(t-s), \quad t \in [s-\delta,s], \quad \xi \in C([-\delta,0];\boldsymbol{R}), \end{cases} \tag{5.24}$$

and the conditions (5.18)–(5.20) hold. A classical (smooth V) verification theorem for this situation was given by Elsanousi and Larssen [5].

Remark 5.2. We may regard the HJB-equation (5.22)–(5.23) as an "effective" equation, since it is the corresponding HJB-equation for the control problem without delay where the system is given by

$$\begin{cases} d\tilde{X} = \Big[\hat{\mu}(\tilde{X}) - g(t,\tilde{X},u)\Big]\,dt + \sigma(\tilde{X})dB, \quad t \in (s,T], \\ \tilde{X}(s) = \tilde{x}, \end{cases} \tag{5.25}$$

with the performance functional

$$J(s,\tilde{x};u) = E^{s,\tilde{x},u}\left[\int_s^T f(t,\tilde{X}(t),u(t))\,dt + h(\tilde{X}(T))\right], \tag{5.26}$$

and the value function

$$\begin{cases} V(s,\tilde{x}) = \sup\limits_{u\in\tilde{\mathcal{U}}[s,T]} J(s,\tilde{x};u), \quad (s,\tilde{x}) \in [0,T) \times \boldsymbol{R}, \\ V(T,\tilde{x}) = h(\tilde{X}(T)), \quad \tilde{x} \in \boldsymbol{R}. \end{cases} \tag{5.27}$$

Since there is no delay, by uniqueness the solution of the HJB-equation (5.27) must be the value function for this control problem.

6. Examples

In this section we present two examples that satisfy the requirements (5.18) – (5.21), and also indicate why it is difficult to find more general examples.

6.1. **Harvesting with exponential growth.** Assume that the size $X(t)$ of a population obeys the linear SDDE

$$\begin{cases} dX = (\mu_1 X + \mu_2 Y + \mu_3 Z - u)\,dt + (\sigma_1 X + \sigma_2 Y)\,dB, \quad t \in [s,T], \\ X(t) = \xi(t-s), \quad t \in [s-\delta,s], \quad \xi \in C([-\delta,0];\boldsymbol{R}). \end{cases} \tag{6.1}$$

We assume that $X(s) > 0$. The population is harvested at a rate $u \geq 0$, and we are given the performance functional

$$J(s,\xi;u) = E^{s,\xi,u}\left[\int_s^T \big\{f_1(X,Y) + f_2(u)\big\}\,dt + h(X(T),Y(T))\right], \tag{6.2}$$

where T is the stopping time

$$T = \min\Big\{t_1, \inf_{t>s}\big\{X^{s,\xi,u}(t) = 0\big\}\Big\}, \tag{6.3}$$

and $t_1 > s$ is some finite deterministic time. If $V = V(s,x,y)$, then V satisfies the HJB equation

$$\begin{aligned} -\partial_s V + \inf_u \Big\{ &-(\mu_1 x + \mu_2 y + \mu_3 z - u)\,\partial_x V - f_1(x,y) - f_2(u) - \frac{1}{2}(\sigma_1 x + \sigma_2 y)^2\,\partial_x^2 V \Big\} \\ &- \big(x - e^{-\lambda\delta}z - \lambda y\big)\,\partial_y V = 0. \end{aligned} \tag{6.4}$$

When will this equation have a solution independent of z? Using Theorem 5.1, from (5.18)–(5.19) we find that the parameters must satisfy the relations

$$\sigma_2 = \sigma_1 \mu_3 e^{\lambda\delta}, \quad \mu_2 - \lambda\mu_3 e^{\lambda\delta} = \mu_3 e^{\lambda\delta}\left(\mu_1 + \mu_3 e^{\lambda\delta}\right). \tag{6.5}$$

The function F defined in (5.13) now has the form

$$F(p,x,y) = \inf_{u\in U}\{pu - f_2(u)\} - f_1(x,y) = p\bar{u} - f_2(\bar{u}) - f_1(x,y),$$

where $\bar{u}$ is an optimal control. Then from (5.20) we see that we must have

$$\partial_y f_1 - \mu_3 e^{\lambda\delta}\partial_x f_1, \tag{6.6}$$

or $f_1 = f_1(x + \mu_3 e^{\lambda\delta} y)$. Introducing the variable $\tilde{x} = x + \mu_3 e^{\lambda\delta} y$ and the constant $\kappa = \mu_1 + \mu_3 \exp(\lambda\delta)$, we find that the "effective" equation (5.22)–(5.23) in this case is

$$-\partial_s V - (\kappa\tilde{x} - \bar{u})\,\partial_{\tilde{x}} V - \frac{1}{2}\sigma_1^2 \tilde{x}^2 \partial_{\tilde{x}}^2 V - f_1(\tilde{x}) - f_2(\bar{u}) = 0, \tag{6.7}$$

with terminal condition

$$V(T,\tilde{x}) = h(\tilde{x}), \tag{6.8}$$

assuming that h satisfies (5.21). This corresponds to the control problem without delay with system dynamics

$$\begin{cases} d\tilde{X} = \left(\kappa\tilde{X} - u\right)dt + \sigma_1 \tilde{X}\,dB, & t \in (s,T], \\ \tilde{X}(s) = \tilde{x}. \end{cases}$$

To close the discussion of this example, let us be specific and choose

$$f_1(x,y) = -c_0\left|x + \mu_3 e^{\lambda\delta} y - m\right|, \quad f_2(u) = c_1 u - c_2 u^2, \quad h = 0, \tag{6.9}$$

where c_0, c_1, c_2, and m are positive constants. Then (6.6) and (5.21) holds and

$$F(p,x,y) = \inf_{u\in U}\left\{c_2 u^2 - (c_1 - p)u\right\} + c_0\left|x + \mu_3 e^{\lambda\delta} y - m\right|.$$

We solve for u and find that the optimal harvesting rate is given by

$$\bar{u} = \max\left\{\frac{c_1 - \partial_x V}{2c_2}, 0\right\}. \tag{6.10}$$

Inserting (6.9) and (6.10) into the HJB-equation (6.7)–(6.8), this may be solved numerically and the optimal control can be found provided the solution of the HJB-equation really is the value function (see Remark 5.1).

6.1.1. *A numerical example.* We now show the numerical solution of the effective equation (6.7). In order to do this we have employed a semi explicit Lax-Friedrichs scheme. Here V_j^n denotes the approximate solution at $t = n\Delta t$, $\tilde{x} = \tilde{x}_j = j\Delta x$, for small discretization parameters Δt and Δx.

$$\begin{aligned} V_j^N &= h(\tilde{x}_j) \\ V_j^{n-1} &= -\frac{1}{2}\left(V_{j-1}^n + V_{j+1}^n\right) + \Delta t F\left(D_j^0 V^n, \tilde{x}_j\right) \\ &\qquad - \frac{1}{2}\sigma_1^2 \tilde{x}_j^2 D_j^+ D_j^- V^{n-1}, \quad 0 \le n \le N, \end{aligned} \tag{6.11}$$

where

$$D_j^0 V = \frac{V_{j+1} - V_{j-1}}{2\Delta x}, \quad D_j^{\pm} V = \pm\frac{V_{j\pm1} - V_j}{\Delta x}.$$

We know from general theory [19] that the sequence of approximate solutions produced by (6.11) converges to the unique viscosity solution of (6.7) if $\Delta t \le \max|\partial_p F|$. In our example we used parameter values

$$\begin{gathered} c_0 = 5, \quad c_1 = 1, \quad c_2 = 2, \quad m = 1, \\ \mu_1 = \mu_2 = \mu_3 = \sigma_1 = 0.5, \\ \lambda = 0.33 \quad \text{and} \quad \delta = 0.25, \end{gathered}$$

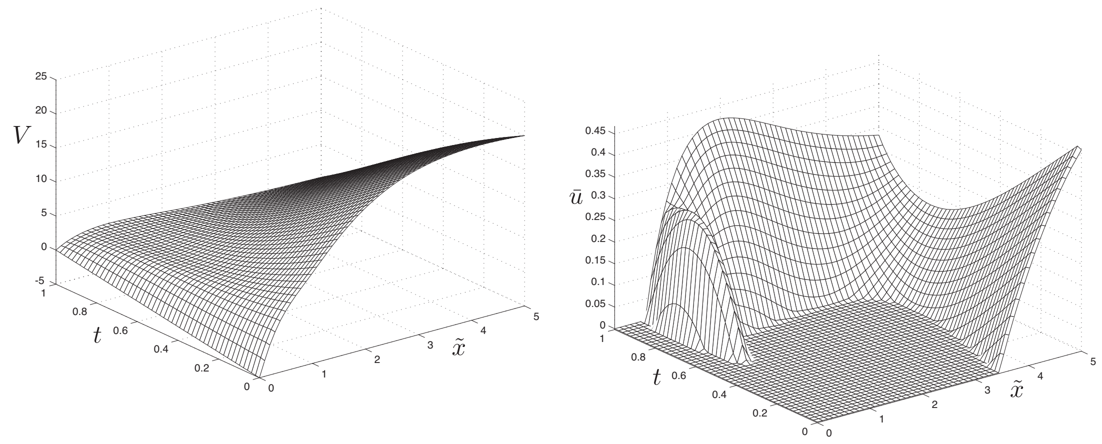

FIGURE 1. The value function V (left) and the optimal harvesting rate $\bar{u}$ (right).

and $h(\tilde{x}) = 5\tilde{x}\exp(-\tilde{x}/m)$. In Figure 1 we show the (numerically calculated) value function V and the optimal harvesting rate $\bar{u}$ as functions of $(t, \tilde{x})$ in the region $[0,1] \times [0,5]$. For $\tilde{x} = 0$, we used boundary values given by $V(t,0) = -(T-t)c_0 m$.

The left pane of Figure 1 indicates that the value function for the effective problem is *smooth*. This means in turn that it must coincide with the value function for the original problem with delay (see Remark 5.1).

6.2. **Allocation with external cost.** Let $X(t)$ denote a population developing according to (6.1). We think of X as a wild population that can be caught and bred in captivity, and then harvested. The population in captivity, $\hat{X}$, developes according to

$$\begin{cases} d\hat{X} = (r\hat{X} + u - v)dt, & t \in (s,T], \\ \hat{X}(s) = \hat{x}, \end{cases} \tag{6.12}$$

where v denotes the harvesting rate. The results in the previous sections for one-dimensional systems obviously generalizes to this case where the system $(X, \hat{X})$ is two-dimensional. The delay enters as before, we only add one extra dimension without delay. For this case we consider the gain functional

$$J(s,\xi,\hat{x};u,v) = E^{s,\xi,\hat{x},u,v}\left[\int_s^T l(v(t)) - c_1\hat{X}(t) - c_2u^2(t)\,dt + h(X(T),Y(T),\hat{X}(T))\right], \tag{6.13}$$

where T is given by (6.3), $l(v)$ denotes the utility from consumption or sales of the animals, $c_1\hat{X}$ models the cost of keeping the population, and c_2u^2 models the cost of catch and transfer. Setting

$$V(s,\xi,\hat{x}) = \sup_{u\geq 0, v} J(s,\xi,\hat{x};u,v),$$

we find that if $V = V(s,x,y,\hat{x})$, then

$$\begin{aligned} -\partial_s V - (\mu_1 x + \mu_2 y + \mu_3 z)\,\partial_x V - r\hat{x}\partial_{\hat{x}}V - \frac{1}{2}(\sigma_1 x + \sigma_2 y + \sigma_3 z)^2\,\partial_x^2 V& \\ - \left(x - e^{-\lambda\delta}z - \lambda y\right)\partial_y V + c_1\hat{x} + F\left(\partial_x V, \partial_{\hat{x}} V\right) &= 0, \\ V(T,x,y,\hat{x}) = h(x,y,\hat{x}).& \end{aligned} \tag{6.14}$$

where

$$F(p,q) = \inf_{\substack{u\geq 0\\ v\leq v_{\max}}} \left(c_2u^2 - u(q-p) + vq - l(v)\right).$$

Since v is independent of z, we must demand that the parameters satisfy (6.5), and introduce $\tilde{x}$ as before, to find that $V = V(s,\tilde{x},\hat{x})$ satisfies

$$-\partial_s V - \kappa\tilde{x}\partial_{\tilde{x}}V - r\hat{x}\partial_{\hat{x}}V - \frac{1}{2}\sigma_1^2\tilde{x}^2\,\partial_{\tilde{x}}^2 V + c_1\hat{x} + F\left(\partial_{\tilde{x}}V,\partial_{\hat{x}}V\right) = 0, \quad \text{for } t < T, \tag{6.15}$$

and $V(T,\tilde{x},\hat{x}) = h(\tilde{x},\hat{x})$. To solve this numerically, we have implemented the two dimensional version of (6.11),

$$\begin{aligned} V_{i,j}^{n-1} &= \frac{1}{4}\left(V_{i,j+1}^n + V_{i,j-1}^n + V_{i+1,j}^n + V_{i-1,j}^n\right) \\ &\quad - \Delta t\Big[-\kappa\tilde{x}_i D_{i,j}^{\tilde{x}}V^n - r\hat{x}_j D_{i,j}^{\hat{x}}V^n + c_1\hat{x}_j + F\left(D_{i,j}^{\hat{x}}V^n, D_{i,j}^{\tilde{x}}V^n\right)\Big] \\ &\quad + \frac{\Delta t}{2}\tilde{x}_i^2\sigma_1^2 D_{i,j}^{+,\tilde{x}}D_{i,j}^{-,\tilde{x}}V^{n-1}, \end{aligned} \tag{6.16}$$

where $V_{i,j}^n$ denotes the approximate solution at $\tilde{x} = \tilde{x}_i = i\Delta\tilde{x}$, $\hat{x} = \hat{x}_j = j\Delta\hat{x}$, and $t = t_n = n\Delta t$. Furthermore,

$$D_{i,j}^{\tilde{x}}V = \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta\tilde{x}}, \quad D_{i,j}^{\hat{x}}V = \frac{V_{i,j+1} - V_{i-1,j}}{2\Delta\tilde{x}} \quad \text{and} \quad D_{i,j}^{\pm,\tilde{x}}V = \pm\frac{V_{i\pm1,j} - V_{i,j}}{\Delta\tilde{x}}.$$

As an example, we have solved (6.16) for t in the interval $[0,1]$ and $(\tilde{x},\hat{x})$ in the quarter plane $\tilde{x} \geq 0$, $\hat{x} \geq 0$ and used boundary values $V(t,\tilde{x},0) = 0$, and

$$h(\tilde{x},\hat{x}) = \tilde{x}\hat{x}\exp\left(1 - 0.25\,\left(\tilde{x}^2 + \hat{x}^2\right)\right), \quad l(v) = 2\sqrt{v}.$$

Furthermore we used the following parameters

$$\begin{gathered} c_1 = 1, \quad c_2 = 2, \quad \mu_1 = 0.7, \quad \mu_2 = \mu_3 = 0.15, \quad \sigma_1 = 0.25, \\ r = 1.5, \quad \lambda = 0.33, \quad \delta = 0.25, \quad \text{and} \quad v_{\max} = 20. \end{gathered}$$

In Figure 2 we show the approximate value function calculated by the above scheme for various times, for $(\tilde{x},\hat{x}) \in [0,5]\times[0,5]$ and with $\Delta\tilde{x} = \Delta\hat{x} = 1/10$.

Figure 2 indicates that the value function for the effective problem in this case is not C^1. Thus we cannot apriori expect it to coincide with the value function for the original problem with delay (see Remark 5.1).

6.3. **Difficulties with nonlinear models.** To use Theorem 5.1 we must be able to solve the equations (5.18)–(5.21). We have seen that for linear systems this was easy. But what about nonlinear systems? Let the (nonlinear) system be given by

$$\begin{cases} dX = \big[\mu(X,Y) + \beta(X,Y)Z - g(u)\big]dt + \sigma(X,Y)dB, & t \in (s,T], \\ X(t) = \xi(t-s), \quad t \in [s-\delta,s], \quad \xi \in C([-\delta,0];\boldsymbol{R}), \end{cases} \tag{6.17}$$

with the gain functional

$$J(s,\xi;u) = E^{s,\xi,u}\left[\int_s^T \left\{f_1(X,Y) + f_2(u)\right\}dt + h(X(T),Y(T))\right]. \tag{6.18}$$

To ease notation we let $\lambda = 0$. If the value function V depends on (s,x,y) only, it solves

$$-\partial_s V - \big[\mu(x,y) + x\beta(x,y)\big]\partial_x V - \frac{1}{2}\sigma^2(x,y)\partial_x^2 V + F\left(\partial_x V, x, y, s\right) = 0. \tag{6.19}$$

When does this equation have a solution independent of z? We consider two cases.

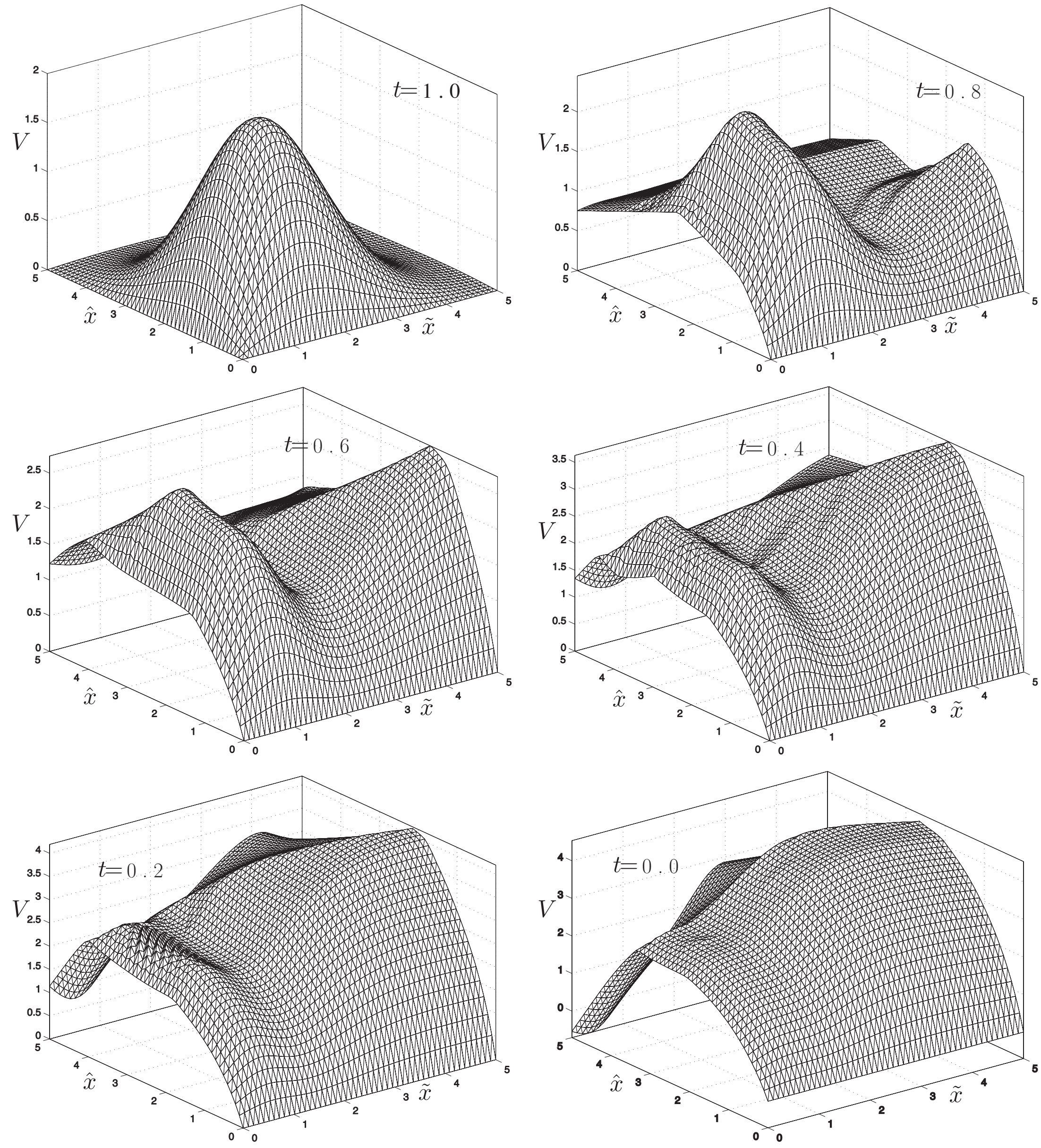

FIGURE 2. The value function $V(\tilde{x}, \hat{x})$ for various times t.

6.3.1. *The case $\beta(x, y) = \beta = constant$.* In this case we have

$$\hat{\mu}(x, y) = \mu(x, y) + \beta x,$$

and equation (5.18) is $\partial_y \hat{\mu} - \beta \partial_x \hat{\mu} = 0$, or equivalently,

$$\hat{\mu} = \hat{\mu}(x + \beta y).$$

Thus we must have

$$\mu(x, y) = \mu(x + \beta y) - \beta x. \tag{6.20}$$

Similar arguments leads to

$$\sigma = \sigma(x + \beta y)$$

by equation (5.19), and

$$f_1 = f_1(x + \beta y) \tag{6.21}$$

by equation (5.20), since $\partial_x \beta = 0$. We conclude that equation (6.19) has a solution independent of z if and only if (6.20)–(6.21) hold, in which case the system takes the form

$$\begin{cases} dX = \big[\mu(X + \beta Y) - \beta \cdot (X - Z) - g(u)\big]dt + \sigma(X + \beta Y)dB, & t \in (s, T], \\ X(t) = \xi(t - s), \quad t \in [s - \delta, s], & \xi \in C([-\delta, 0]; \boldsymbol{R}). \end{cases} \tag{6.22}$$

6.3.2. *The case $\beta(x, y) = \beta x$.* In this case we have

$$\hat{\mu}(x, y) = \mu(x, y) + \beta x^2,$$

and equation (5.18) is $\partial_y \hat{\mu} - \beta x \partial_x \hat{\mu} = 0$, or equivalently,

$$\hat{\mu} = \hat{\mu}(xe^{\beta y}).$$

Thus we must have

$$\mu(x, y) = \mu(xe^{\beta y}) - \beta x^2, \tag{6.23}$$

and similarly,

$$\sigma = \sigma(xe^{\beta y})$$

by equation (5.19). To solve equation (5.20), we note first that

$$\begin{aligned} F(p, x, y) &= \inf_{u \in U} \big\{ g(u)p - f_2(u) \big\} - f_1(x, y) \\ &= g(\bar{u})p - f_2(\bar{u}) - f_1(x, y), \end{aligned}$$

where $\bar{u} = \bar{u}(p)$ is an optimal control. Also note that we must have $g'(\bar{u}) - f_2'(\bar{u}) = 0$. Thus,

$$\begin{aligned} \partial_p F &= g'(\bar{u}) \cdot \bar{u}_p + g(\bar{u}) - f_2'(\bar{u}) \cdot \bar{u}_p \\ &= \big(g'(\bar{u}) - f_2'(\bar{u})\big) \cdot \bar{u}_p + g(\bar{u}) \\ &= g(\bar{u}). \end{aligned}$$

So equation (5.20) is

$$pg(\bar{u})\beta - \big(\partial_y f_1 - \beta x \partial_x f_1\big) = 0. \tag{6.24}$$

The first term in this equation is a function of p only, the second term is a function of (x, y) only. Hence both terms must be zero. This can happen only when $\beta = 0$. But then our model collapses to a model without delay.

This indicates that when $\beta(x, y)$ is not constant, in order to obtain a nontrivial reduced nonlinear delay model, one may have to consider coefficients depending on the control in a more complicated way than in the system (6.17). This makes for less mathematical tractability as the corresponding equations of type (5.19)–(5.21) will be more complicated.

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